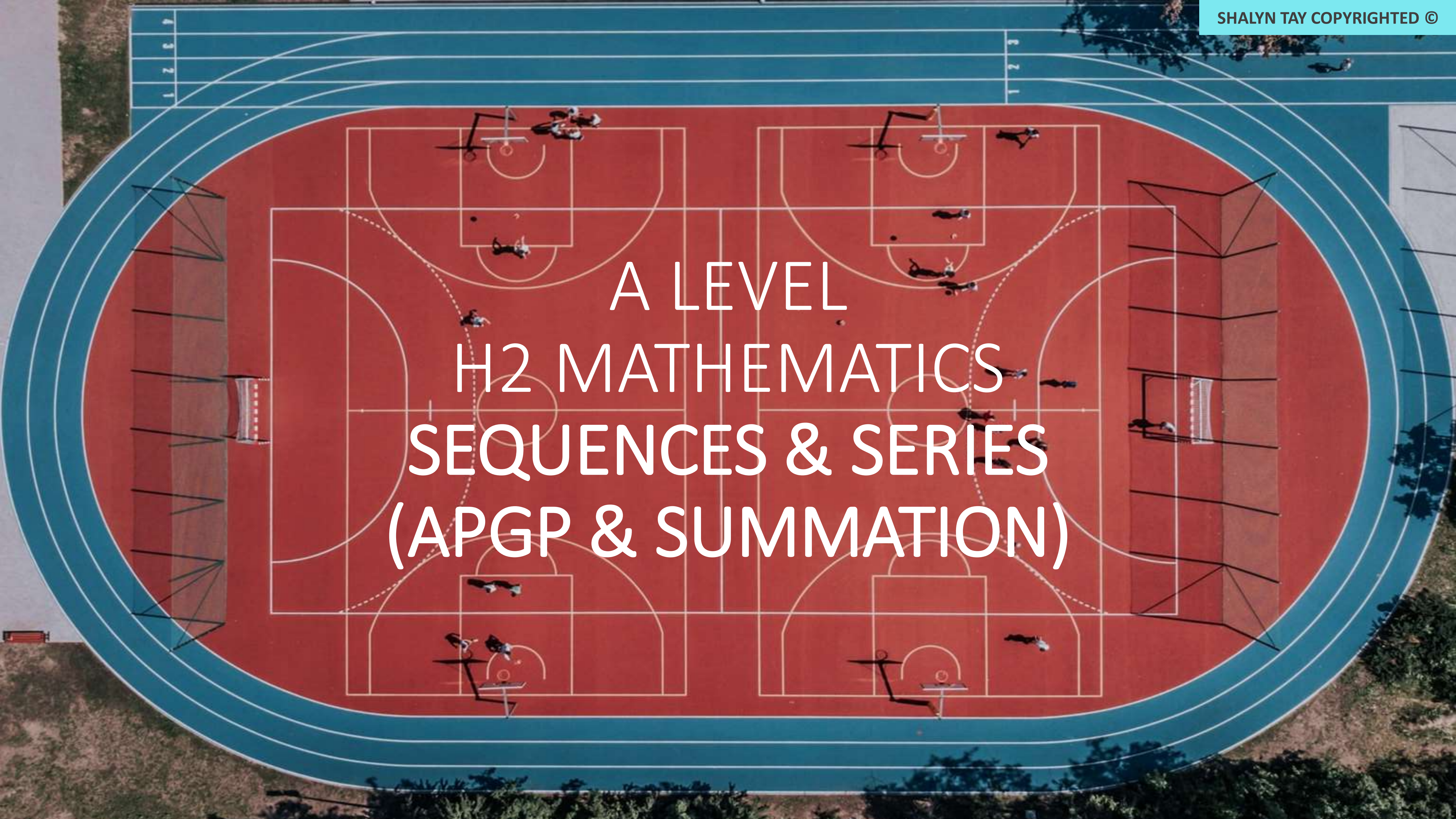

A LEVEL
H2 MATHEMATICS
SEQUENCES & SERIES (APGP & SUMMATION)

CHAPTER ANALYSIS

MASTERY

- Concept of Sequences and Series
- Arithmetic Progression
- Geometric Progression
- Sigma Notation and Summation of Series
- Method of Difference
- Convergence and Sum to Infinity

EXAM

- Useful to be familiar with partial fractions
- Practice contextual questions for APGP
- Many students struggle with solving summation questions in which the starting term is not 1

WEIGHTAGE

- Appears every year, 1-2 questions
- Typically constitutes approximately 7-8% of final grade

Sequence

$$u_1, u_2, u_3, \ldots, u_n$$

The entire set of ordered elements is a **sequence**

Term of the sequence is usually denoted by $\boldsymbol{u_n}$

Series

$$S_n = u_1 + u_2 + u_3 + \ldots + u_n$$

Sum of a Sequence

Sequences & Series

A **sequence** is a set of numbers arranged in a defined order. Each number in the sequence is called a **term** of the sequence, u_n

Types of Sequences:

- Finite: sequence terminates
- Infinite: there is no end, sequence keeps going on…
- Convergent: u_n approaches a unique value (the limit) as $n \to \infty$
- Divergent: sequence that does not converge
- Strictly increasing: $u_{n+1} > u_n$ for all n
- Strictly decreasing: $u_{n+1} < u_n$ for all n
- Constant: $u_n = c$ for all n

A **series** is the sum of the terms of a sequence, denoted by S_n . It can have a finite or infinite number of terms.

Useful results:

$$u_n = S_n - S_{n-1} \text{ for } n > 1$$

$$u_1 = S_1$$

SEQUENCES & SERIES PART I

ARITHMETIC PROGRESSION
GEOMETRIC PROGRESSION

Proof for Arithmetic Progressions

To prove a sequence is an AP, show:

$$u_n - u_{n-1} = constant\ for\ all\ values\ of\ n > 1$$

(common difference, *d*)

"Since the difference between 2 consecutive terms is a constant, the terms of this sequence form an arithmetic progression."

"Since $u_3 - u_2 \neq u_2 - u_1$ the terms of the sequence do not form an arithmetic progression."

Arithmetic Progression (AP)

An **arithmetic progression** is a sequence of numbers for which the **difference** between every 2 consecutive terms is a **constant**, known as the common difference, *d*:

$$u_n - u_{n-1} = d$$

The ***n*th term** of the arithmetic progression is:

Term of AP

$$u_n = a + (n-1)\, d$$

Common Difference

First Term

Sum of first *n* terms **of an arithmetic series** is:

$$S_n = \frac{n}{2}[2a + (n-1)\, d] = \frac{n}{2}\,(u_1 + u_n)$$

Proof for Geometric Progressions

To prove a sequence is an GP, show:

$$\frac{u_n}{u_{n-1}} = constant\ for\ all\ values\ of\ n > 1$$

(common ratio, *r*)

"Since the ratio of every 2 consecutive terms is a constant, the terms of this sequence form a geometric progression."

"Since $\frac{u_3}{u_2} \neq \frac{u_2}{u_1}$ *the terms of the sequence do not form a geometric progression."*

Geometric Progression (GP)

An **<u>geometric progression</u>** is a sequence of numbers for which the **ratio** of every 2 consecutive terms is a **constant**, known as the common ratio, *r*:

$$\frac{u_n}{u_{n-1}} = r$$

The ***n*th term** of the geometric progression is:

Term of GP

$$u_n = ar^{n-1}$$

Common Ratio

First Term

Sum of first *n* terms **of a geometric series** is:

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r}, r \neq 1$$

Sum to infinity **of a geometric series**:

$$S_\infty = \frac{a}{1 - r}, |r| < 1$$

SEQUENCES & SERIES PART I

SUMMATION OF SERIES
METHOD OF DIFFERENCE

Summation of Series

Standard Results

$$\sum_{r=1}^{n} a = na$$

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2 = \left(\frac{1}{2}n(n+1)\right)^2 = \left(\sum_{r=1}^{n} r\right)^2$$

Arithmetic Progression

$$\sum_{r=1}^{n}[a+(r-1)d] = \frac{n}{2}[2a+(n-1)d]$$

Geometric Progression

$$\sum_{k=1}^{n} ar^{k-1} = \frac{a(r^n-1)}{r-1}$$

The Sigma Notation:

$$\sum_{r=k}^{n} u_r = u_k + u_{k+1} + \cdots + u_{n-1} + u_n$$

n is the upper limit of the summation or the ending value of r

When the series is infinite, n is replaced with ∞

r is a dummy variable known as the index of summation

k is the lower limit of the summation or the starting value of r

A series has $(\boldsymbol{n}-\boldsymbol{k}+\mathbf{1})$ terms

E.g. $$\sum_{r=1}^{5}\frac{1}{3^r} = \frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}$$

You may also use the G.C. to find the sum of a series directly [math > summation]

Properties of Sigma Notation:

$$\sum_{r=k}^{n} au_r \pm bv_r = a\sum_{r=k}^{n} u_r \pm b\sum_{r=k}^{n} v_r$$

where a and b are constants

$$\sum_{r=k}^{n} u_r = \sum_{r=1}^{n} u_r - \sum_{r=k}^{k-1} u_r$$

for $2 \leq k \leq n$

Convergence / Divergence of Series

Typically, we can also denote $\sum_{r=1}^{n} u_r$ by S_n .

If S_n approaches a **unique** value as *n* approaches infinity, we say S_n **converges** and the sum to infinity S_∞ exists.

Or else it is said to **diverge**.

$$\sum_{r=1}^{n} \frac{1}{r} - \frac{1}{r+1}$$

$$\begin{aligned}
&= \left(\frac{1}{1} - \frac{1}{2}\right) \\
&+ \left(\frac{1}{2} - \frac{1}{3}\right) \\
&+ \left(\frac{1}{3} - \frac{1}{4}\right) \\
&\quad \vdots \\
&+ \left(\frac{1}{n-2} - \frac{1}{n-1}\right) \\
&+ \left(\frac{1}{n-1} - \frac{1}{n}\right) \\
&+ \left(\frac{1}{n} - \frac{1}{n-1}\right) \\
&= \left(\frac{1}{1} - \frac{1}{n-1}\right)
\end{aligned}$$

Geometric Series:

$$\boxed{S_\infty = \frac{a}{1-r}, \ |r| < 1}$$

Sum to Infinity:

As n approaches ∞ , $\frac{1}{n-1} \to 0$,

$$\sum_{r=1}^{\infty} \frac{1}{r} - \frac{1}{r+1} \to 1 - 0 = 1$$

Method of Difference

The method of difference is used to evaluate sums where the term u_r can be expressed as a **difference** of two or more terms which will result in the **cancellation** of most terms leaving only a final solution.

$$\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} (f(r) - f(r-1))$$

$$\begin{aligned}
&= (f(1) - f(0)) \\
&+ (f(2) - f(1)) \\
&+ (f(3) - f(2)) \\
&\quad \vdots \\
&+ (f(n-2) - f(n-3)) \\
&+ (f(n-1) - f(n-2)) \\
&+ (f(n) - f(n-1)) \\
&= (f(n) - f(0))
\end{aligned}$$

Many questions usually require the use of **partial fractions** to express the term u_r into the difference of more terms.