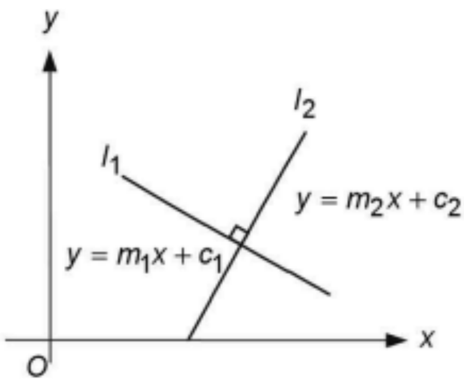
SECONDARY 3 ADDITIONAL MATHEMATICS
Coordinate Geometry, Equations of Circles, Trigonometry

Name:________________________________

Date: ________________________________

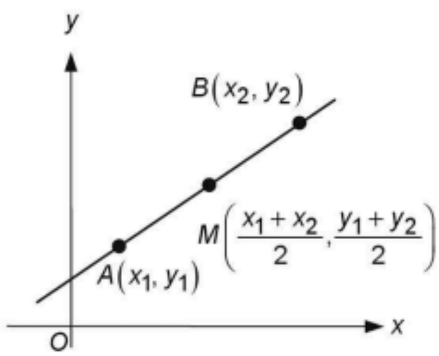
Coordinate Geometry

1. Gradient of perpendicular line



- If two lines are perpendicular, then the product of their gradients is -1.

2. Coordinates of midpoint



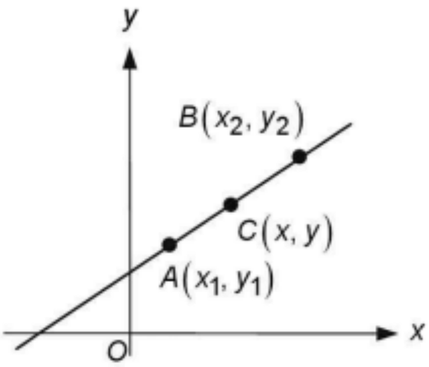
- The coordinates of the midpoint $M(x, y)$, of two points $A(x_1, y_1)$ and $B(x_2, y_2)$, is given by finding the average of two points.
- This can be also found by the formula

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

- **Tip! The midpoint can be used to find the fourth vertex of a parallelogram since the two diagonals share the same midpoint.**

3. Perpendicular bisector
   - Combining points #1 and #2 above, we can get the gradient of a line's perpendicular bisector (using #1), as well as a point it passes through (#2).
   - This allows us to find the equation of a perpendicular bisector.

4. Collinear points



If three points, A, B and C are collinear, then

Gradient of AB = Gradient of BC = Gradient of AC

5. Area of rectilinear figure

If $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3), \ldots$ and $M(x_m, y_m)$ are arranged in an anti-clockwise manner, the area of rectilinear figure can be found by

$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \ldots & x_m & x_1 \\ y_1 & y_2 & y_3 \ldots & y_m & y_1 \end{vmatrix}$$

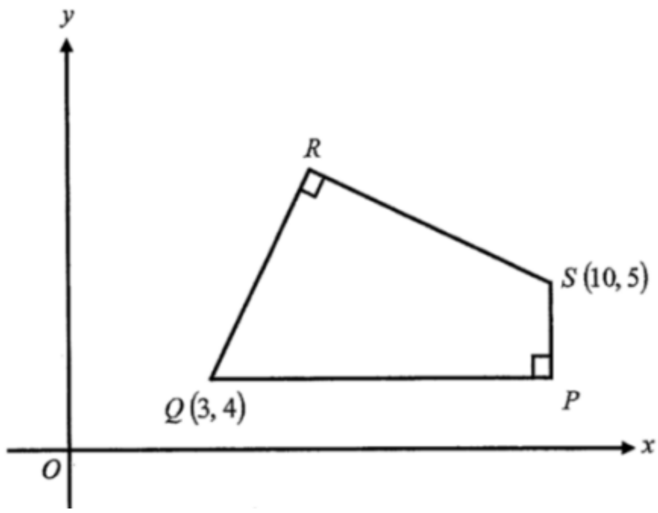
$$= \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \ldots & x_m & x_1 \\ y_1 & y_2 & y_3 \ldots & y_m & y_1 \end{vmatrix}$$

$$= \frac{1}{2}\left|[x_1y_2 + x_2y_3 + \cdots + x_my_1] - [y_1x_2 + y_2x_3 + \cdots + y_mx_1]\right|$$

- The bars on the side of the formula are **<u>not</u>** modulus signs. They are a way of representing a concept called "determinants".
- If the coordinates represented in the formula is arranged in a **clockwise** manner instead, the resultant area will be **negative.**

Worked Example 1

Solutions to this question by accurate drawing will not be accepted.



The diagram shows a quadrilateral $PQRS$ in which SR is perpendicular to RQ and QP is perpendicular to PS. The point Q is $(3, 4)$ and the point S is $(10, 5)$. Given that QR is parallel to the line $6x - 2y = 13$, find

(a) The equation of QR

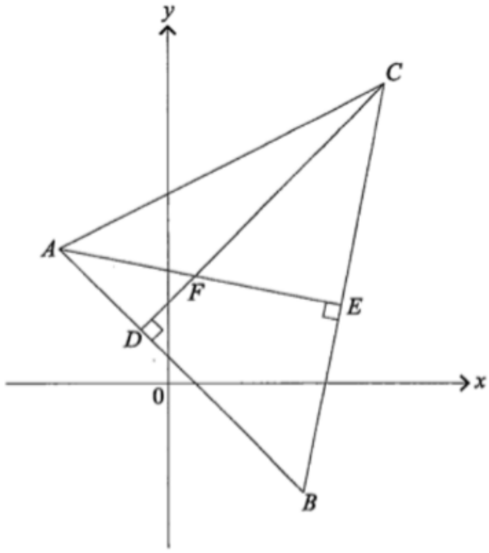
(b) The coordinates of R

(c) The area of the quadrilateral $PQRS$

(d) T is a point on the line SR such that the area of ΔQTR : area of $\Delta QTS = 3 : 2$. Find the coordinates of the point T.

[S3 YCKSS P1/2018 EOY Qn 11]

Worked Example 2



The vertices of the triangle ABC have coordinates $(-4, 5), (5, -4)$ and $(8, 11)$ respectively. AE is perpendicular to BC, CD is perpendicular to AB, and CD and AE meets at F.

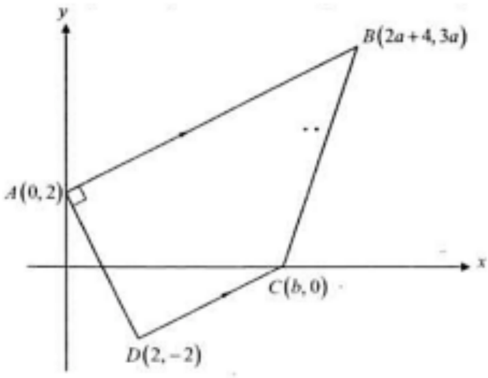
(a) Find the coordinates of D and of F.

(b) Find the area of ΔABC

[S3 BSS P2/2018 EOY Qn 10]

Worked Example 3

Solutions to this question by accurate drawing will not be accepted.



The diagram, not drawn to scale, shows a trapezium $ABCD$ in which AB is parallel to DC and $\angle BAD = 90°$. The vertices of the trapezium are the points$A(0,2)$, $B(2a+4, 3a)$, $C(b, 0)$ and $D(2, -2)$

(a) Given that the length of AB is $4\sqrt{5}$ units, find the value of a, where $a > 0$

(b) Find the equation of AB

(c) Find the value of b

(d) Find the perpendicular bisector of AB

(e) Hence, or otherwise, show that C lies on perpendicular bisector of AB

(f) Find the area of the trapezium $ABCD$

[S4 NGHS P2/2015 PRELIM Qn 9]

Equations of Circles

We can express the equation of circles in two forms.

	Standard	General
Equation	$(x-h)^2+(y-k)^2=r^2$	$x^2+y^2+2gx+2fy+c=0$
Centre	(h,k)	$(-g,-f)$
Radius	r	$\sqrt{g^2+f^2-c}$

We can obtain the standard form of the equation of circle by completing the square from the general form:

$$\begin{aligned} x^2+y^2+2gx+2fy+c &= x^2+2gx+y^2+2fy+c \\ &= x^2+2gx+g^2+y^2+2fy+f^2+c-f^2-g^2 \\ &= (x+g)^2+(y+f)^2+c-f^2-g^2 \end{aligned}$$

Hence, the general equation of circle can be algebraically manipulated into

$$(x+g)^2+(y+f)^2=g^2+f^2-c$$

which gives us the centre and radius of the circle as shown in the table above.

Position of point relative to circle

To test/determine if a point is inside, on or outside of a circle, we can use the compare the length of the radius with the length between the point and the centre of the circle:

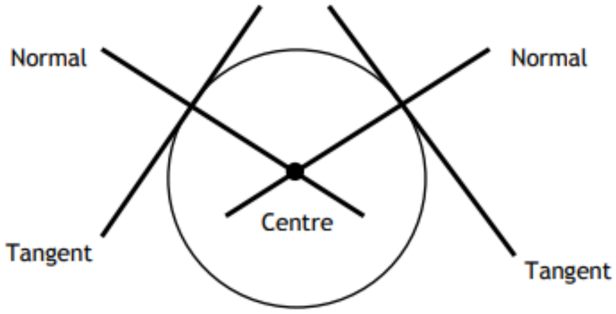
Test	Result
Inside the circle	Radius > Length
On the circle	Radius = Length
Outside the circle	Radius < Length

Note: Another test to determine if a point is <u>on the circle</u> is to substitute the point into the equation of the circle. If the equation holds, the point lies on the circle.

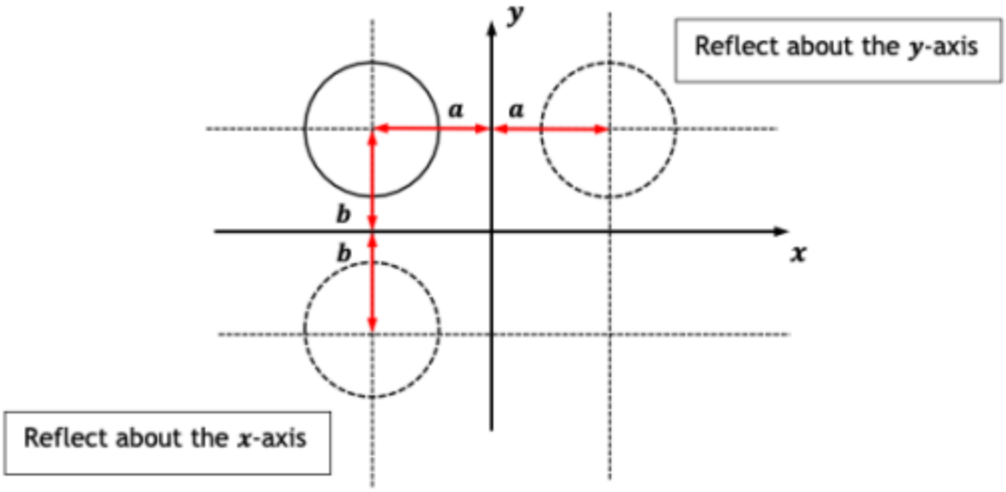
Note

The concept of **normal and perpendicular bisectors** is usually tested in this chapter. Usually the equation of tangent, and the coordinate it touches the circle will be given, allowing us to find the equation of normal easily.

- The normal passes through the centre of the circle.
- Any two normals of a circle intersects at the centre of the circle.



Reflections

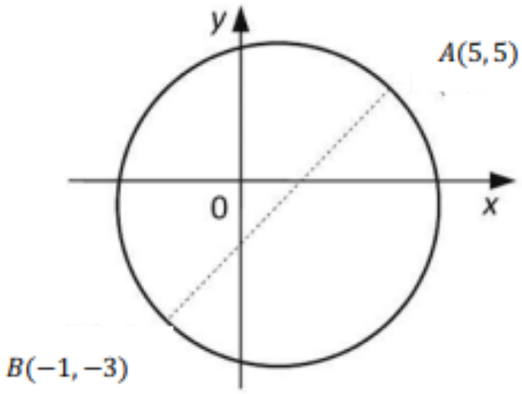


Note!

- Radius of the circle remains constant.
- Distance between the centre of the original circle to the axis remains constant.
- For the centre, when reflecting about the x-axis, the x-coordinate does not change. ONLY y-coordinate changes.
- For the centre, when reflecting about the y-axis, the y –coordinate does not change. ONLY x-coordinate changes.

Worked Example 4

Solutions to this question by accurate drawing will not be accepted.



In the diagram above, the points $A(5,5)$ and $B(-1,-3)$ are opposite ends of a diameter of a circle.

(a) Find the equation of the circle in its general form.

(b) If $C(6, k)$ lies on the circle, find the value of k, where $k < 0$.

(c) Hence, prove that ΔABC is an isosceles right-angled triangle.

Worked Example 5

The equation of a circle, C_1, is $x^2 + y^2 - 4x + 8y + 11 = 0$

(a) Find the coordinates of the centre of C_1 and its radius.

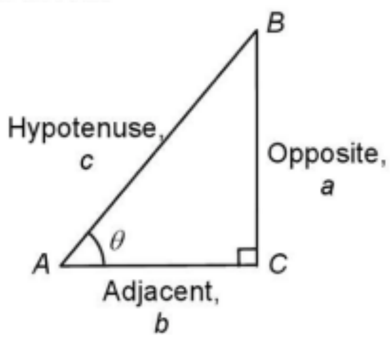
(b) Find the equation of the circle, C_2 , which is a reflection of C_1 about the line $y = 1$.

(c) Given that the line $y - x + 3 = 0$ intersects the circle , C_1, at two points R and S, find the equation of the perpendicular bisector of the line RS.

[S4 AHS P2/2021 PRELIM Qn 10]

Introduction to Trigonometry

1. Trigonometry Ratios of Acute Angles (TOA CAH SOH)



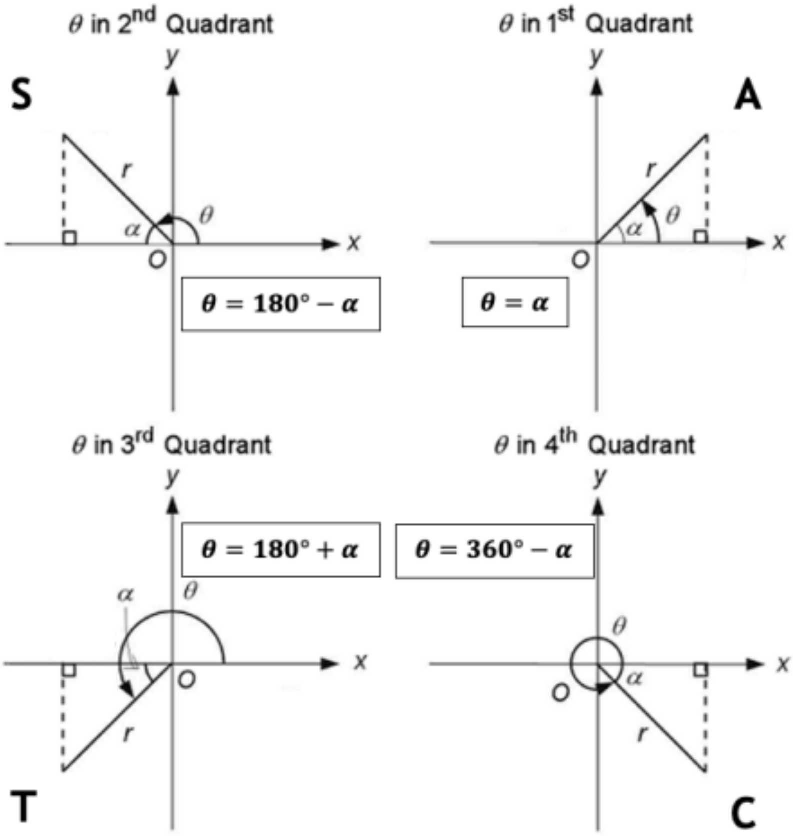
Ratio	Formula
sine	$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}$
cosine	$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}$
tangent	$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b}$

2. Special Angles

θ	0^o	30^o	45^o	60^o	90^o
	0 rad	$\frac{1}{6}\pi$ rad	$\frac{1}{4}\pi$ rad	$\frac{1}{3}\pi$ rad	$\frac{1}{2}\pi$ rad
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

Note! You do not need to memorise these values! The trick is to key into the calculator and if you see a decimal, square the value and you will get the fraction. Afterwards, apply a square root to the fraction and rationalize!

3. Four Quadrants **(ASTC)**



4. Reciprocal Ratios

5. **Ratio**	**Formula**
secant	$\sec\theta = \dfrac{1}{\cos\theta}$
cosecant	$\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$
cotangent	$\cot\theta = \dfrac{1}{\tan\theta}$

Note! The trick here is to look out for the third letter! E.g. If the third letter is "c", it means it is 1 / Cosine.

5. Trigonometric Identities

Identity	Formula
Quotient	$\tan A = \dfrac{\sin A}{\cos A}$
	$\cot A = \dfrac{\cos A}{\sin A}$
Reciprocal	$\sec A = \dfrac{1}{\cos A}$
	$cosec\ A = \dfrac{1}{\sin A}$
	$\cot A = \dfrac{1}{\tan A}$
Pythagorean	$\sin^2 A + \cos^2 A = 1^*$
	$\sec^2 A = 1 + \tan^2 A^*$
	$\text{cosec}^2 A = 1 + \cot^2 A^*$
Addition Formula	$\sin(A \pm B) = \sin A\ cosB\ \pm \cos A \sin B^*$
	$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B^*$
	$\tan(A \pm B) = \dfrac{\tan A\ \pm \tan B}{1 \mp \tan A \tan B}$
Double Angle Formula (gives us Half Angle Formula)	$\sin 2A = 2 \sin A \cos A^*$
	$\cos 2A = \cos^2 A - \sin^2 A^*$ $= 2\cos^2 A - 1^*$ $= 1 - 2\sin^2 A^*$
	$\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}\ ^*$

Note! Formulas with the asterisk are provided in your formula sheet!

Additional Tips!

- $\mathbf{\sin(-x) = -\sin x}$
- $\mathbf{\cos(-x) = \cos x}$
- $\mathbf{\tan(-x) = -\tan x}$

Worked Example 1

Given that $\mathbf{\tan A = -\frac{12}{5}}$ and that $\mathbf{\tan A}$ $\boldsymbol{and}$ $\mathbf{\cos A}$ have opposite signs, find the value of

(a) $\mathbf{\cos(-A)}$

(b) $\mathbf{\sin A \cos A}$

(c) $\mathbf{\cos 2A - \sin 2A}$

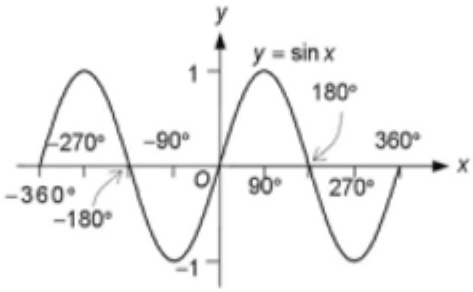
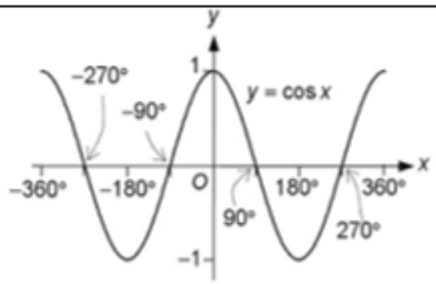
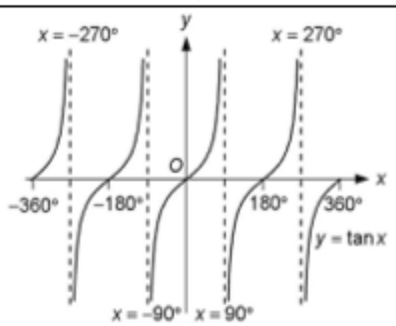
Worked Example 2

Given that $\mathbf{\sin A = -\frac{4}{5}}$, $\mathbf{\tan B = -\frac{5}{12}}$ and $\mathbf{\cos A > 0}$, and that $\boldsymbol{A}$ and $\boldsymbol{B}$ are in different quadrants, evaluate the following without the use of a calculator,

(a) $\mathbf{\cot A}$

(b) $\mathbf{\sin(\frac{B}{2})}$

Graphs of Trigonometric Functions

Trigonometric Ratio	Graphical Representation	Features
$y = \sin\theta$		Amplitude = 1 Period = 360° or 2π
$y = \cos\theta$		Amplitude = 1 Period = 360° or 2π
$y = \tan\theta$		Period = 180° or π

Take Note

- For tangent graphs, the dotted lines represent vertical asymptotes. At these vertical asymptotes, the graph approaches very close to it but never touches /crosses them.
- ***a***
  - $|a|$ is the amplitude of the graph
  - Distance of the highest / lowest points from the equilibrium position
  - $a < 0$ implies the graph is flipped (reflected about the ***x***-axis)
- ***b***
  - Determines period of the graph (how long the graph completes 1 cycle)
  - $b = \frac{360^o}{\text{period}}$
- ***c***
  - How much the equilibrium line has shifted up or shifted down from the origin. (translation)

Steps to plot any trigonometric graphs :

1) Identify the amplitude and period of the graph
2) Plot your x and y axis
3) Take note of the range of the graph and label down the x- values for one complete cycle (period). For this portion, split up the period into 5 different $x - values$.
4) Substitute the values of x identified in Step 3 into the equation to find the corresponding y values.
5) Connect the dots with a smooth curve and repeat the steps above if needed.

Worked Example 3

Given the two equations $y = 3\sin x - 1$ and $y = \tan\frac{x}{2}$,

(a) Sketch the two graphs of the two equations for the range $0° \leq x \leq 360°$.

(b) Hence, state the number of solutions for $0° \leq x \leq 360°$, for the equation of $3\sin x - 1 = \tan\frac{x}{2}$

Worked Example 4

It is given that $\boldsymbol{y = 4\cos\frac{x}{3} - 2}$.

(a) State the maximum and minimum value of $\boldsymbol{y}$.

(b) State the amplitude and period of $\boldsymbol{y}$.

(c) Sketch the graph of $\boldsymbol{y = 4\cos\frac{x}{3} - 2}$ for $\boldsymbol{0° \leq x \leq 540°}$

Trigonometric Identities and Equations

- In this topic, you will be tasked to convert trigonometric equations from one form to another (typically from LHS to RHS or vice versa). You are supposed to make use of the trigonometric formulas as shown below to help you with proving trigonometric identities.

Worked Example 5

Prove that

$$\frac{\mathbf{1} + \sin \boldsymbol{A}}{\mathbf{1} - \sin \boldsymbol{A}} = (\tan \boldsymbol{A} + \sec \boldsymbol{A})^2$$

Tip! If you are unsure of which side to start off with for trigonometric identities questions, always start off with the most complex-looking side. It can come in the form of fractions, squares and product of two brackets.
Always remind yourself what you want to achieve on the other side and work from there. For example, if you need to have sine function, make use of the trigonometric formulas and convert what you have into sine functions.

Worked Example 6

Solve the equation $2\cos 2x = 4 + 5\cos x$ for $0° \leq x \leq 360°$.

Step 1: Rearrange the equation with the help of trigonometric identities such that there is only one function.

Step 2: When you have converted the equation into the form above, replace my trigonometric function with another letter and solve it like a quadratic equation. (*It will not always be a quadratic equation*)

Note! Since the value of $\boldsymbol{cos\ x}$ cannot be greater than 1 and lesser than -1, $\boldsymbol{cos\ x = 2}$ will be REJECTED!

Step 3: Find the basic acute angle and identify the quadrants that the possible angles lie in, depending on whether the trigonometric function is positive or negative.

Note! As the name "Basic Acute Angle" suggests, the angle is acute, hence it lies in the first quadrant. Everything in the first quadrant is positive, hence even if the trigonometric function is negative, ALWAYS use the positive value for the basic acute angle!

Step 3 (Continued)

Step 4: Check for the range provided in the question to determine if you have to reject any solutions or if you can turn another round!

Worked Example 7

(a) Prove that $\frac{\tan x}{1-\cot x}+\frac{\cot x}{1-\tan x}=1+\sec x\ \operatorname{cosec} x$.

(b) Hence, without the use of a calculator, find the exact value of $\frac{\tan 75°}{1-\cot 75°}+\frac{\cot 75°}{1-\tan 75°}$